

ASYMPTOTIC BEHAVIOR OF GRADIENT-LIKE DYNAMICAL SYSTEMS INVOLVING INERTIA AND MULTISCALE ASPECTS

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Abstract. In a Hilbert space \mathcal{H} , we study the asymptotic behaviour, as time variable t goes to $+\infty$, of nonautonomous gradient-like dynamical systems involving inertia and multiscale features. Given $\Phi : \mathcal{H} \rightarrow \mathbb{R}$ and $\Psi : \mathcal{H} \rightarrow \mathbb{R}$ two convex differentiable functions, γ a positive damping parameter, and $\epsilon(t)$ a function of t which tends to zero as t goes to $+\infty$, we consider the second-order differential equation

$$\ddot{x}(t) + \gamma \dot{x}(t) + \nabla \Phi(x(t)) + \epsilon(t) \nabla \Psi(x(t)) = 0.$$

This system models the emergence of various collective behaviors in game theory, as well as the asymptotic control of coupled nonlinear oscillators. Assuming that $\epsilon(t)$ tends to zero moderately slowly as t goes to infinity, we show that the trajectories converge weakly in \mathcal{H} . The limiting equilibria are solutions of the hierarchical minimization problem which consists in minimizing Ψ over the set C of minimizers of Φ . As key assumptions, we suppose that $\int_0^{+\infty} \epsilon(t) dt = +\infty$ and that, for every p belonging to a convex cone \mathcal{C} depending on the data Φ and Ψ

$$\int_0^{+\infty} [\Phi^*(\epsilon(t)p) - \sigma_C(\epsilon(t)p)] dt < +\infty$$

where Φ^* is the Fenchel conjugate of Φ , and σ_C is the support function of C . An application is given to coupled oscillators.

Key words: asymptotic behaviour; asymptotic control; convex minimization; hierarchical minimization; inertial dynamic; linear damping; Lyapunov analysis; nonautonomous gradient-like systems; second-order differential equations; slow control; time multiscaling.

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1. INTRODUCTION

\mathcal{H} is a real Hilbert space, we write $\|x\|^2 = \langle x, x \rangle$ for $x \in \mathcal{H}$. For any differentiable function $F : \mathcal{H} \rightarrow \mathbb{R}$, its gradient is denoted by ∇F . Thus $F'(x)(y) = \langle \nabla F(x), y \rangle$. The first order (respectively second order) derivative at time t of a function $x(\cdot) : [0, +\infty[\rightarrow \mathcal{H}$ is denoted by $\dot{x}(t)$ (respectively $\ddot{x}(t)$). Throughout the paper γ is a fixed positive parameter (viscous damping coefficient).

1.1. Problem statement. Henceforth, we make the following standing assumptions (\mathcal{H}_0) on data Φ , Ψ and $\epsilon(\cdot)$, that will be needed throughout the paper. We write $S = \operatorname{argmin}_C \Psi$ with $C = \operatorname{argmin} \Phi$, or, equivalently, $S = \operatorname{argmin} \{\Psi | \operatorname{argmin} \Phi\}$.

$$(\mathcal{H}_0) \quad \begin{cases} \Phi : \mathcal{H} \rightarrow \mathbb{R} \text{ is convex, } \nabla \Phi \text{ is Lipschitz continuous on bounded sets of } \mathcal{H}, C = \operatorname{argmin} \Phi = \Phi^{-1}(0) \neq \emptyset; \\ \Psi : \mathcal{H} \rightarrow \mathbb{R} \text{ is convex, } \nabla \Psi \text{ is Lipschitz continuous on bounded sets of } \mathcal{H}, \Psi \text{ is bounded from below;} \\ S = \operatorname{argmin} \{\Psi | \operatorname{argmin} \Phi\} \neq \emptyset; \\ \epsilon(\cdot) : [0, +\infty[\rightarrow]0, +\infty[\text{ is a nonincreasing function, of class } C^1, \epsilon(t) \text{ and } \dot{\epsilon}(t) \text{ tend to zero as } t \rightarrow +\infty. \end{cases}$$

We study the asymptotic behavior ($t \rightarrow +\infty$) of the trajectories of the nonautonomous Multiscaled Inertial Gradient-like system ((MIG) for short)

$$(\text{MIG}) \quad \ddot{x}(t) + \gamma \dot{x}(t) + \nabla \Phi(x(t)) + \epsilon(t) \nabla \Psi(x(t)) = 0.$$

Let us observe that when $\Psi \equiv 0$, or $\epsilon = 0$, the above dynamic reduces to

$$(\text{HBF}) \quad \ddot{x}(t) + \gamma \dot{x}(t) + \nabla \Phi(x(t)) = 0.$$

The Heavy Ball with Friction dynamical system, (HBF) for short, plays an important role in mechanics, control theory, and optimization, see [1], [6], [10], [23], [27] for a general presentation. Because of the viscosity friction term $\gamma \dot{x}$, it is a dissipative dynamical system (the global mechanical energy is decreasing), which gives it remarkable properties of optimization. Indeed, just assuming that Φ is convex and $C = \operatorname{argmin} \Phi \neq \emptyset$, each trajectory of system (HBF) converges weakly in \mathcal{H} , with a limit belonging to the closed convex set C , see Alvarez [1].

In many situations, there is a continuum of equilibria (equivalently, C is not reduced to a singleton), as it occurs when considering a potential Φ which is not strictly convex. A situation frequently encountered results from the application of a least squares method to an under determined linear problem. Another important situation that we address in this paper concerns the weakly coupled systems. In this paper, we address the question: how to control (HBF) to obtain asymptotically an equilibrium having desirable properties? A somewhat related issue is: why specific equilibria are observed in real life, despite the fact that no force is present who can explain?

We give an answer to these questions by introducing in (HBF) an asymptotically vanishing term $\epsilon(t) \nabla \Psi(x(t))$, so obtaining (MIG). The crucial point is to take a control variable $\epsilon(t)$ which tends to zero in a moderate way, that is, not too fast

$$\int_0^{+\infty} \epsilon(t) dt = +\infty,$$

and not too slow: we suppose that the following inequality holds for every p belonging to a cone, whose definition involves Φ and Ψ ,

$$\int_0^{+\infty} [\Phi^*(\epsilon(t)p) - \sigma_C(\epsilon(t)p)] dt < +\infty,$$

where Φ^* is the Fenchel conjugate of Φ , and σ_C is the support function of C . In this case, we will prove the following asymptotical hierarchical selection property: each trajectory of system (MIG) converges weakly, with a limit belonging to $\operatorname{argmin}_C \Psi$, i.e., the limit does minimize Ψ over the set of minimizers of Φ :

$$(1) \quad x(t) \rightharpoonup x_\infty \in \operatorname{argmin}_C \Psi \text{ as } t \rightarrow +\infty.$$

The case $\Psi(x) = \|x\|^2$, which corresponds to Tikhonov regularization, has been examined by Attouch and Czarnecki in [8]. In this case, each trajectory of (MIG) converges strongly to the unique element of C with minimal norm. A natural extension of this result to the case of a uniformly convex function Ψ was obtained by Cabot in [18].

For many applications, it is important to go beyond the strongly and uniformly convex cases. A typical situation, that we illustrate in Section 5, is the modeling of the (weak) coupling between two oscillators $x(\cdot)$ and $y(\cdot)$, where we take $\Psi(x, y) = \|Ax - By\|^2$, A and B linear continuous operators. We also give an application of this model to the Nash equilibration for potential games, and to coupled wave equations.

The asymptotic analysis of the multiscaled first-order differential inclusion

$$\dot{x}(t) + \partial \Phi(x(t)) + \epsilon(t) \partial \Psi(x(t)) \ni 0,$$

involving general convex potentials Φ and Ψ has been studied by Attouch and Czarnecki in [9]. As we shall see, several asymptotic convergence properties can be carried from the first order to the second order differential system. In particular, the moderate decrease condition on $\epsilon(\cdot)$ which allows to obtain an asymptotic hierarchical minimization property, and which involves the Fenchel conjugate of Φ is quite similar. Note that, by contrast with the first-order case where general convex lower semicontinuous potentials are considered, our analysis of the inertial second-order

differential system deals with differentiable potentials. As a general rule, introducing nonsmooth potentials in inertial systems leads us to consider trajectories whose acceleration is a vectorial measure, and with possible shocks, an interesting topic whose study goes beyond the scope of this article, see [6] for the (HBF) system.

The local Lipschitz continuity assumption on the gradient of the potentials Φ and Ψ ensures the existence of strong solutions to (MIG), but this assumption is not used in the proof of convergence, suggesting some extensions to PDE's. An illustrative example is given in Section 6 concerning coupled wave equations. In this case, the potential Φ (Dirichlet energy) is only lsc. on $\mathcal{H} = L^2(\Omega)$. Still, the existence can be proved by direct energy methods, and we show that the result of convergence holds.

1.2. Moderate decrease condition on $\epsilon(\cdot)$. To prove the convergence of trajectories of (MIG) system, and show the asymptotic hierarchical minimization property (1), we need to make assumptions about how $\epsilon(\cdot)$ tends to zero as t goes to infinity. These conditions express that $\epsilon(\cdot)$ is not too small (\mathcal{H}_1), and not too large (\mathcal{H}_2):

$$(\mathcal{H}_1) \quad \int_0^{+\infty} \epsilon(t) dt = +\infty;$$

$$(\mathcal{H}_2) \quad \forall p \in \mathcal{C} \quad \int_0^{+\infty} [\Phi^*(\epsilon(t)p) - \sigma_C(\epsilon(t)p)] dt < +\infty,$$

with

$$(2) \quad \mathcal{C} = \{\lambda p : \lambda \geq 0, \exists z \in C, p = -\nabla \Psi(z), \nabla \Psi(z) + N_C(z) \ni 0\}.$$

In (\mathcal{H}_2) we use classical concepts and notations of convex analysis: Φ^* is the Fenchel conjugate of Φ , defined for every $y \in \mathcal{H}$ by

$$\Phi^*(y) = \sup \{\langle y, x \rangle - \Phi(x) | x \in \mathcal{H}\},$$

σ_C is the support function of $C = \operatorname{argmin} \Phi$, defined for every $y \in \mathcal{H}$, by

$$\sigma_C(y) = \sup \{\langle y, x \rangle | x \in C\},$$

and $N_C(x)$ is the (outwards) normal cone to C at x .

The cone \mathcal{C} can be equivalently defined as

$$\mathcal{C} = \{\lambda p : \lambda \geq 0, \exists z \in \operatorname{argmin}_C \Psi, p = -\nabla \Psi(z)\}.$$

One can verify that, under some general qualification assumption, \mathcal{C} is a closed convex cone. Indeed

$$\mathcal{C} = \{\lambda p : \lambda \geq 0, p \in \operatorname{argmin}_{\mathcal{H}} (\sigma_C(\cdot) + \Psi^*(-\cdot))\}.$$

Analysis of condition (\mathcal{H}_1) : the slow parametrization condition $\int_0^{+\infty} \epsilon(t) dt = +\infty$ expresses that the control variable $\epsilon(\cdot)$ converges slowly to zero, when time t goes to infinity. It plays a central role in the asymptotic hierarchical property. This fact has been well established in a series of papers concerning first or second order systems of this type, see [7], [8], [9], [17], [18].

In contrast, the fast parametrization case $\int_0^{+\infty} \epsilon(t) dt < +\infty$, behaves exactly as if $\epsilon \equiv 0$. In this case, there is no asymptotic hierarchical property associated to the control variable.

Analysis of condition (\mathcal{H}_2) :

- a) Note that Φ enters in (MIG) only via its subdifferential. Thus it is not a restriction to assume $\min_{\mathcal{H}} \Phi = 0$. For a function Φ whose minimum is not equal to zero, one should replace in (\mathcal{H}_2) and in the corresponding statements Φ by $\Phi - \min_{\mathcal{H}} \Phi$.
From $\Phi \leq \delta_C$ we get $\Phi^* \geq (\delta_C)^* = \sigma_C$, and $\Phi^* - \sigma_C \geq 0$.
We have used that σ_C is equal to the Fenchel conjugate of δ_C , where δ_C is the indicator function of C .
Hence, (\mathcal{H}_2) means that, for $p \in \mathcal{C}$, the nonnegative function

$$t \mapsto [\Phi^*(\epsilon(t)p) - \sigma_C(\epsilon(t)p)]$$

is integrable on $[0, +\infty[$. Practically, this condition forces $\epsilon(\cdot)$ to be small enough for large t .

- b) As an illustration, consider the model situation: $\Phi(z) = \frac{1}{2} \operatorname{dist}^2(z, C) = \frac{1}{2} \|\cdot\|^2 +_e \delta_C$, where $+_e$ denotes the epigraphical sum (also called inf-convolution). From general properties of Fenchel transform

$$\Phi^*(z) = \frac{1}{2} \|z\|^2 + \sigma_C(z) \text{ and } \Phi^*(z) - \sigma_C(z) = \frac{1}{2} \|z\|^2.$$

Hence, in this situation

$$(\mathcal{H}_2) \iff \int_0^{+\infty} \epsilon(t)^2 dt < +\infty.$$

Conditions (\mathcal{H}_1) and (\mathcal{H}_2) are equivalent to $\epsilon(\cdot) \in L^2(0, +\infty) \setminus L^1(0, +\infty)$. It is satisfied for example by taking $\epsilon(t) = \frac{1}{1+t^\alpha}$ with $\frac{1}{2} < \alpha \leq 1$.

1.3. Contents. In Section 2, we express our main result, which shows that, under conditions of moderate decrease of $\epsilon(\cdot)$, each trajectory of (MIG) converges weakly to the solution of a hierarchical minimization problem. In Section 3, we complement this result by showing a property of strong convergence when Ψ is strongly convex. Then, in Section 4, we look at some multiscale aspects, and show the effect of changing the scaling in dynamic. In particular, we make the link between asymptotic vanishing viscosity and penalization. Finally, we illustrate our study with applications to coupled oscillators in Section 5, and coupled wave equations in Section 6.

2. WEAK CONVERGENCE RESULTS

In this section, we consider the dynamical system

$$(MIG) \quad \ddot{x}(t) + \gamma \dot{x}(t) + \nabla \Phi(x(t)) + \epsilon(t) \nabla \Psi(x(t)) = 0.$$

When $\Psi = 0$, (MIG) boils down to the classical second order damped nonlinear oscillator

$$(HBF) \quad \ddot{x}(t) + \gamma \dot{x}(t) + \nabla \Phi(x(t)) = 0.$$

In accordance with Alvarez theorem [1], and Attouch-Czarnecki [8], in our main result, which is stated below, we are going to show that, under moderate decrease condition on $\epsilon(\cdot)$, each trajectory of (MIG) converges weakly in \mathcal{H} to a minimizer of Φ , which also minimizes Ψ over all minima of Φ .

Theorem 2.1. *Let us suppose that standing assumptions (\mathcal{H}_0) are satisfied:*

$$(\mathcal{H}_0) \quad \left\{ \begin{array}{l} \Phi : \mathcal{H} \rightarrow \mathbb{R} \text{ is convex, } \nabla \Phi \text{ is Lipschitz continuous on bounded sets of } \mathcal{H}, C = \operatorname{argmin} \Phi = \Phi^{-1}(0) \neq \emptyset; \\ \Psi : \mathcal{H} \rightarrow \mathbb{R} \text{ is convex, } \nabla \Psi \text{ is Lipschitz continuous on bounded sets of } \mathcal{H}, \Psi \text{ is bounded from below;} \\ S = \operatorname{argmin} \{ \Psi | \operatorname{argmin} \Phi \} \neq \emptyset; \\ \epsilon(\cdot) : [0, +\infty[\rightarrow]0, +\infty[\text{ is a nonincreasing function, of class } C^1, \epsilon(t) \text{ and } \dot{\epsilon}(t) \text{ tend to zero as } t \rightarrow +\infty. \end{array} \right.$$

Let x be a classical maximal solution of (MIG). Then $x(\cdot)$ is defined on $[0, +\infty[$. Let us assume moreover that the moderate decrease conditions on $\epsilon(\cdot)$ are satisfied:

$$\begin{aligned} (\mathcal{H}_1) \quad & \int_0^{+\infty} \epsilon(t) dt = +\infty; \\ (\mathcal{H}_2) \quad & \forall p \in \mathcal{C} \quad \int_0^{+\infty} [\Phi^*(\epsilon(t)p) - \sigma_C(\epsilon(t)p)] dt < +\infty, \\ & \text{with } \mathcal{C} = \{ \lambda p : \lambda \geq 0, \exists z \in C, p = -\nabla \Psi(z), \nabla \Psi(z) + N_C(z) \ni 0 \}; \\ (\mathcal{H}_3) \quad & 0 \leq -\dot{\epsilon}(t) \leq k\epsilon^2(t) \quad \text{for some positive constant } k, \text{ and large enough } t. \end{aligned}$$

Then

$$\begin{aligned} (i) \quad & \text{weak convergence} \quad \exists x_\infty \in S = \operatorname{argmin} \{ \Psi | \operatorname{argmin} \Phi \}, \text{ such that } w - \lim_{t \rightarrow +\infty} x(t) = x_\infty; \\ (ii) \quad & \text{minimizing properties} \quad \lim_{t \rightarrow +\infty} \Phi(x(t)) = 0 \text{ and } \lim_{t \rightarrow +\infty} \Psi(x(t)) = \min \Psi|_{\operatorname{argmin} \Phi}. \end{aligned}$$

Taking $\Psi = 0$ in Theorem 2.1, we recover the Alvarez convergence result for the heavy ball with friction system.

Corollary 2.1. [1, Theorem 2.1] *Let $\Phi : \mathcal{H} \rightarrow \mathbb{R}$ be a convex function whose gradient $\nabla \Phi$ is Lipschitz continuous on the bounded subsets of \mathcal{H} , and such that $C = \operatorname{argmin} \Phi \neq \emptyset$. Let $\gamma > 0$ be a positive damping parameter. Then for any solution trajectory $x(\cdot)$ of*

$$(HBF) \quad \ddot{x}(t) + \gamma \dot{x}(t) + \nabla \Phi(x(t)) = 0,$$

$x(t)$ converges weakly in \mathcal{H} to a point in $\operatorname{argmin} \Phi$, as t goes to $+\infty$.

Proof of corollary 2.1. Take $\Psi \equiv 0$, and choose $\epsilon(\cdot)$ an arbitrary function satisfying the standing assumptions, (\mathcal{H}_1) and (\mathcal{H}_3) (for example take $\epsilon(t) = \frac{1}{1+t}$). We claim that (\mathcal{H}_2) is automatically satisfied. Since $\Psi \equiv 0$, the cone \mathcal{C} is reduced to the origin. Noticing that $\Phi^*(0) = -\inf_{\mathcal{H}} \Phi = 0$, and that $\sigma_C(0) = 0$, we have $\int_0^{+\infty} [\Phi^*(\epsilon(t)0) - \sigma_C(\epsilon(t)0)] dt = 0$. Thus (\mathcal{H}_2) is satisfied, and we can apply Theorem 2.1. As a conclusion, we obtain the weak convergence of $x(t)$ to a point in $\operatorname{argmin} \Phi$, as t goes to $+\infty$ (since Ψ is constant, there is no hierarchical minimization associated to it). That's the conclusion of Alvarez theorem.

Remark 2.1. The counterexample of Baillon [12] shows that one may not have strong convergence for (HBF). Of course, the same holds for (MIG).

Proof of Theorem 2.1. As in Bruck [16], weak convergence is a consequence of Opial's lemma, after showing the convergence of $\|x(\cdot) - z\|$ for every $z \in S$, and that every weak cluster point of x belongs to S . The proof consists of three steps, each of these steps relying on a different Lyapunov function.

Step 1: First energy estimates and global existence results. First note that, for any Cauchy data $x(0) = x_0$, $\dot{x}(0) = \dot{x}_0$, with $x_0 \in \mathcal{H}$, $\dot{x}_0 \in \mathcal{H}$, local existence and uniqueness of the corresponding classical solution to (MIG) is a direct consequence of Cauchy-Lipschitz theorem, after reducing (MIG) to a first order system.

Let $x : [0, T[\rightarrow \mathcal{H}$ be a maximal solution of (MIG). To prove that $T = +\infty$ we use a classical contradiction argument. Let us suppose that $T < +\infty$. To obtain a contradiction, it is enough to show that $\lim_{t \rightarrow T} x(t)$ and $\lim_{t \rightarrow T} \dot{x}(t)$ exist. This is a consequence of the energy estimates which are described below, which show that $\|\dot{x}(t)\|$ and $\|\ddot{x}(t)\|$ are bounded on $[0, T[$. In order to avoid repeating twice the same argument, we establish these estimations directly on global solutions of (MIG), with $T = +\infty$.

Given $x(\cdot) : [0, +\infty[\rightarrow \mathcal{H}$ a global classical solution of (MIG), set

$$(3) \quad E_1(t) = \frac{1}{2} \|\dot{x}(t)\|^2 + \Phi(x(t)) + \epsilon(t)\Psi(x(t)).$$

We are going to show that E_1 is a Lyapunov-like function, which will allow us to derive several asymptotic properties of trajectories of (MIG).

Lemma 2.1. *Let us just assume that standing assumptions (\mathcal{H}_0) hold. Then, for any trajectory $x(\cdot) : [0, +\infty[\rightarrow \mathcal{H}$ of (MIG), the following properties hold:*

- (i) $t \mapsto E_1(t) - c\epsilon(t)$ is a decreasing function, with $c = \inf_{\mathcal{H}} \Psi$;
- (ii) $\dot{x} \in L^2([0, +\infty[; \mathcal{H}) \cap L^\infty([0, +\infty[; \mathcal{H})$.

Assuming moreover that $x(\cdot)$ is bounded, then

- (iii) $\lim_{t \rightarrow +\infty} \dot{x}(t) = \lim_{t \rightarrow +\infty} \ddot{x}(t) = 0$;
- (iv) $\lim_{t \rightarrow +\infty} \nabla \Phi(x(t)) = 0$, and hence $\lim_{t \rightarrow +\infty} \Phi(x(t)) = \inf_{\mathcal{H}} \Phi$.

Proof of Lemma 2.1. Let us compute the time derivative of $E_1(\cdot)$. Using the classical derivation chain rule and equation (MIG), we obtain

$$\begin{aligned} \dot{E}_1(t) &= \langle \dot{x}(t), \ddot{x}(t) \rangle + \langle \nabla \Phi(x(t)), \dot{x}(t) \rangle + \epsilon(t) \langle \nabla \Psi(x(t)), \dot{x}(t) \rangle + \dot{\epsilon}(t) \Psi(x(t)) \\ &= \langle \dot{x}(t), \ddot{x}(t) + \nabla \Phi(x(t)) + \epsilon(t) \nabla \Psi(x(t)) \rangle + \dot{\epsilon}(t) \Psi(x(t)) \\ &= -\gamma \|\dot{x}(t)\|^2 + \dot{\epsilon}(t) \Psi(x(t)). \end{aligned}$$

Using that $\epsilon(\cdot)$ is a decreasing function (i.e., $\dot{\epsilon}(t) \leq 0$), and that Ψ is bounded from below, we deduce that

$$\dot{E}_1(t) + \gamma \|\dot{x}(t)\|^2 \leq c \dot{\epsilon}(t),$$

with $c = \inf_{\mathcal{H}} \Psi$, that's item (i). After integration of this inequality with respect to t , we get

$$(4) \quad \begin{aligned} E_1(t) - E_1(0) + \gamma \int_0^t \|\dot{x}(s)\|^2 ds &\leq c(\epsilon(t) - \epsilon(0)) \\ &\leq |c| \epsilon(0). \end{aligned}$$

As a consequence

$$\sup_{t \geq 0} E_1(t) < +\infty.$$

By definition (3) of E_1 , we infer the existence of some constant C_1 such that for all $t \geq 0$

$$\frac{1}{2} \|\dot{x}(t)\|^2 + \Phi(x(t)) + \epsilon(t) \Psi(x(t)) \leq C_1.$$

Since Φ and Ψ are bounded from below and $\epsilon(\cdot)$ is bounded, it follows that

$$(5) \quad \sup_{t \geq 0} \|\dot{x}(t)\| < +\infty.$$

Noticing that $E_1(\cdot)$ is bounded from below, inequality (4) also yields

$$(6) \quad \int_0^{+\infty} \|\dot{x}(t)\|^2 dt < +\infty.$$

Collecting (5) and (6), we obtain item (ii).

Let us now suppose that $x(\cdot)$ is bounded, i.e.,

$$\sup_{t \geq 0} \|x(t)\| < +\infty.$$

By standing assumptions (\mathcal{H}_0) , $\nabla \Phi$ and $\nabla \Psi$ are Lipschitz continuous on bounded sets of \mathcal{H} , and hence bounded on bounded sets of \mathcal{H} . From equation (MIG), also using the boundedness of $\epsilon(\cdot)$, and the boundedness of $\dot{x}(\cdot)$ (item (ii)), it follows that

$$\ddot{x}(\cdot) \in L^\infty([0, +\infty); \mathcal{H}).$$

Hence $\dot{x}(\cdot)$ is Lipschitz continuous on $]0, +\infty[$, a property which combined with $\dot{x} \in L^2([0, +\infty[; \mathcal{H})$ classically implies

$$(7) \quad \lim_{t \rightarrow +\infty} \dot{x}(t) = 0.$$

Let us now prove that

$$(8) \quad \lim_{t \rightarrow +\infty} \ddot{x}(t) = 0.$$

For the sake of simplicity, we assume that Φ and Ψ are twice differentiable (otherwise, the argument can be justified by using finite difference quotients). After derivation of equation (MIG) we obtain

$$\ddot{x}(t) + \gamma \dot{x}(t) + \nabla^2 \Phi(x(t)) \dot{x}(t) + \epsilon(t) \nabla^2 \Psi(x(t)) \dot{x}(t) + \dot{\epsilon}(t) \nabla \Psi(x(t)) = 0.$$

The Lipschitz continuity on bounded sets of $\nabla \Phi$ and $\nabla \Psi$, together with (7) imply

$$\lim_{t \rightarrow +\infty} \nabla^2 \Phi(x(t)) \dot{x}(t) = \lim_{t \rightarrow +\infty} \nabla^2 \Psi(x(t)) \dot{x}(t) = 0.$$

Using too that $\lim_{t \rightarrow +\infty} \epsilon(t) = \lim_{t \rightarrow +\infty} \dot{\epsilon}(t) = 0$, we obtain

$$(9) \quad \ddot{x}(t) + \gamma \dot{x}(t) = g(t)$$

for some function g such that $\lim_{t \rightarrow +\infty} g(t) = 0$. Integration of (9) yields

$$\dot{x}(t) = e^{-\gamma t} \dot{x}(0) + e^{-\gamma t} \int_0^t e^{\gamma s} g(s) ds,$$

which, by $\lim_{t \rightarrow +\infty} g(t) = 0$, and an elementary argument, gives (8).

Let us now return to (MIG) equation to obtain

$$(10) \quad \lim_{t \rightarrow +\infty} \nabla \Phi(x(t)) = 0.$$

By using a standard convexity argument we are going to deduce that

$$(11) \quad \lim_{t \rightarrow +\infty} \Phi(x(t)) = \inf_{\mathcal{H}} \Phi.$$

To that end, let us write the convex differential inequality at an arbitrary $\xi \in \mathcal{H}$

$$\Phi(\xi) \geq \Phi(x(t)) + \langle \nabla \Phi(x(t)), \xi - x(t) \rangle.$$

Using that $x(\cdot)$ is bounded and (10) we immediately obtain

$$\Phi(\xi) \geq \limsup_{t \rightarrow +\infty} \Phi(x(t)) \geq \liminf_{t \rightarrow +\infty} \Phi(x(t)) \geq \inf_{\mathcal{H}} \Phi.$$

This inequality being true for any $\xi \in \mathcal{H}$, we obtain (11).

Step 2: Using distance to equilibria as a Lyapunov function. For an element $z \in S = \operatorname{argmin}\{\Psi | \operatorname{argmin} \Phi\}$, we define the function $h_z : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ by

$$h_z(t) = \frac{1}{2} \|x(t) - z\|^2.$$

First write the optimality condition for $z \in S$. Recall that $C = \operatorname{argmin} \Phi$. Since $z \in \operatorname{argmin}(\Psi + \delta_C)$, we have $0 \in \nabla \Psi(z) + \partial \delta_C(z)$. Since $\delta_C(z) = N_C(z)$,

$$0 \in \nabla \Psi(z) + N_C(z).$$

Equivalently,

$$(12) \quad \exists p \in N_C(z) \text{ such that } -p = \nabla \Psi(z).$$

Using h_z as a Lyapunov-like function provides further informations that are described in the following lemma.

Lemma 2.2. *Let us assume that standing assumptions (\mathcal{H}_0) hold together with the moderate decrease properties (\mathcal{H}_1) and (\mathcal{H}_2) . Then, for any trajectory $x(\cdot) : [0, +\infty[\rightarrow \mathcal{H}$ of (MIG), the following properties hold:*

- (i) for any $z \in S$, $\lim_{t \rightarrow +\infty} h_z(t)$ exists;
- (ii) $\int_0^{+\infty} \Phi(x(t)) dt < +\infty$;
- (iii) $\liminf_{t \rightarrow +\infty} \Psi(x(t)) \geq \Psi(z)$.

Proof of Lemma 2.2. Given $z \in S$, let us compute the first and second order time derivatives of $h_z(\cdot)$. Using the classical derivation chain rule, we obtain

$$(13) \quad \dot{h}_z(t) = \langle x(t) - z, \dot{x}(t) \rangle$$

$$(14) \quad \ddot{h}_z(t) = \langle x(t) - z, \ddot{x}(t) \rangle + \|\dot{x}(t)\|^2.$$

By using (MIG) equation it follows that

$$\begin{aligned} \ddot{h}_z(t) + \gamma \dot{h}_z(t) &= \langle x(t) - z, \ddot{x}(t) + \gamma \dot{x}(t) \rangle + \|\dot{x}(t)\|^2 \\ &= \langle x(t) - z, -\nabla \Phi(x(t)) - \epsilon(t) \nabla \Psi(x(t)) \rangle + \|\dot{x}(t)\|^2. \end{aligned}$$

Equivalently

$$(15) \quad \ddot{h}_z(t) + \gamma \dot{h}_z(t) + \langle \nabla \Phi(x(t)), x(t) - z \rangle + \epsilon(t) \langle \nabla \Psi(x(t)), x(t) - z \rangle = \|\dot{x}(t)\|^2.$$

Let us now write the convex differential inequalities (recall that $z \in \operatorname{argmin} \Phi$, and hence $\Phi(z) = 0$)

$$(16) \quad 0 = \Phi(z) \geq \Phi(x(t)) + \langle \nabla \Phi(x(t)), z - x(t) \rangle,$$

$$(17) \quad \Psi(z) \geq \Psi(x(t)) + \langle \nabla \Psi(x(t)), z - x(t) \rangle.$$

Combining (15), (16) and (17), we obtain

$$(18) \quad \ddot{h}_z(t) + \gamma \dot{h}_z(t) + \Phi(x(t)) + \epsilon(t) (\Psi(x(t)) - \Psi(z)) \leq \|\dot{x}(t)\|^2.$$

Let us now use the optimality condition (12), i.e., $-p = \nabla \Psi(z)$ with $p \in N_C(z)$. We have

$$(19) \quad \Psi(x(t)) \geq \Psi(z) + \langle -p, x(t) - z \rangle.$$

Combining (18) and (19)

$$(20) \quad \ddot{h}_z(t) + \gamma \dot{h}_z(t) \leq (\epsilon(t) \langle p, x(t) \rangle - \Phi(x(t))) - \epsilon(t) \langle p, z \rangle + \|\dot{x}(t)\|^2.$$

By definition of the Fenchel conjugate

$$(21) \quad \epsilon(t) \langle p, x(t) \rangle - \Phi(x(t)) \leq \Phi^*(\epsilon(t)p).$$

On the other hand, by $p \in N_C(z)$, we have

$$(22) \quad \epsilon(t) \langle p, z \rangle = \sigma_C(\epsilon(t)p).$$

Combining (21), (22) with (20) we finally obtain

$$(23) \quad \ddot{h}_z(t) + \gamma \dot{h}_z(t) \leq [\Phi^*(\epsilon(t)p) - \sigma_C(\epsilon(t)p)] + \|\dot{x}(t)\|^2.$$

By (12), p belongs to the cone \mathcal{C} as defined in (\mathcal{H}_2) . Hence, by assumption (\mathcal{H}_2) we have that

$$\int_0^{+\infty} [\Phi^*(\epsilon(t)p) - \sigma_C(\epsilon(t)p)] dt < +\infty.$$

On the other hand, we know by Lemma 2.1 that $\int_0^{+\infty} \|\dot{x}(t)\|^2 dt < +\infty$.

Hence the second member of (23) is integrable on $[0, +\infty[$, and the convergence of $h_z(t)$ as $t \rightarrow +\infty$ is a consequence of the following lemma, see [1, Lemma 2.2 and 2.3], [8, Lemma 3.1].

Lemma 2.3. *Let $t_0 \in \mathbb{R}$ and $h \in \mathcal{C}^2([t_0, +\infty[, \mathbb{R}^+)$ satisfy the following differential inequality*

$$\ddot{h}(t) + \gamma \dot{h}(t) \leq g(t)$$

with $g \in L^1([t_0, +\infty[, \mathbb{R}^+)$. Then $(\dot{h})_+$, the positive part of \dot{h} , belongs to $L^1([t_0, +\infty[, \mathbb{R})$, and, as a consequence, $\lim_{t \rightarrow +\infty} h(t)$ exists.

Thus item (i) of Lemma 2.2 is proved. In order to obtain further estimates, let us return to (18). The same computation as above relying on the definition of the Fenchel conjugate yields

$$(24) \quad -[\Phi^*(\epsilon(t)p) - \sigma_C(\epsilon(t)p)] \leq \Phi(x(t)) + \epsilon(t) (\Psi(x(t)) - \Psi(z)).$$

Combining (18) with (24), we obtain

$$\ddot{h}_z(t) + \gamma \dot{h}_z(t) - [\Phi^*(\epsilon(t)p) - \sigma_C(\epsilon(t)p)] \leq \ddot{h}_z(t) + \gamma \dot{h}_z(t) + \Phi(x(t)) + \epsilon(t) (\Psi(x(t)) - \Psi(z)) \leq \|\dot{x}(t)\|^2.$$

Let us integrate the above inequalities between two large real numbers. Using that $\lim_{t \rightarrow +\infty} h(t)$ exists, as it has been just proved, noticing $\lim_{t \rightarrow +\infty} \dot{h}_z(t) = 0$ (this last point results from (13), the convergence of $\dot{x}(t)$ to zero, and the boundedness of $x(\cdot)$), and using assumption (\mathcal{H}_2) , we conclude that

$$(25) \quad \lim_{t \rightarrow +\infty} \int_0^t [\Phi(x(s)) + \epsilon(s) (\Psi(x(s)) - \Psi(z))] ds \text{ exists.}$$

Let us now use the same device as [9], and split in equation (20) the term $\Phi(x(t))$ into two parts, so obtaining

$$(26) \quad \begin{aligned} \ddot{h}_z(t) + \gamma \dot{h}_z(t) + \frac{1}{2} \Phi(x(t)) &\leq \|\dot{x}(t)\|^2 + \frac{1}{2} [\langle 2\epsilon(t)p, x(t) \rangle - \Phi(x(t))] - \frac{1}{2} \langle 2\epsilon(t)p, z \rangle \\ &\leq \|\dot{x}(t)\|^2 + \frac{1}{2} [\Phi^*(2\epsilon(t)p) - \sigma_C(2\epsilon(t)p)]. \end{aligned}$$

Since \mathcal{C} is a cone, $2p$ still belongs to \mathcal{C} , which by assumption (\mathcal{H}_2) yields integrability of the second member of inequality (26). It follows at once that

$$(27) \quad \int_0^{+\infty} \Phi(x(t)) dt < +\infty,$$

which proves item (ii). Combining (25) and (27), we also get

$$\lim_{t \rightarrow +\infty} \int_0^t \epsilon(s) (\Psi(x(s)) - \Psi(z)) ds \text{ exists.}$$

Using assumption (\mathcal{H}_1) , i.e., $\int_0^{+\infty} \epsilon(t) dt = +\infty$, we obtain item (iii), that is

$$\liminf_{t \rightarrow +\infty} \Psi(x(t)) \leq \Psi(z).$$

Step 3: Using a rescaled energy as a Lyapunov function. Detecting the property of asymptotic hierarchical minimization requires further Lyapunov analysis based on the rescaled energy function

$$E_2(t) = \frac{1}{2\epsilon(t)} \|\dot{x}(t)\|^2 + \frac{1}{\epsilon(t)} \Phi(x(t)) + \Psi(x(t)).$$

Lemma 2.4. *Let us assume that standing assumptions (\mathcal{H}_0) hold together with the moderate decrease properties (\mathcal{H}_1) , (\mathcal{H}_2) and (\mathcal{H}_3) . Then, for any trajectory $x(\cdot) : [0, +\infty[\rightarrow \mathcal{H}$ of (MIG), and any $z \in S$, the following properties hold:*

- (i) $\dot{E}_2(t) = -\frac{\gamma}{\epsilon(t)} \|\dot{x}(t)\|^2 - \frac{\dot{\epsilon}(t)}{\epsilon(t)^2} \left(\frac{1}{2} \|\dot{x}(t)\|^2 + \Phi(x(t)) \right);$
- (ii) $\lim_{t \rightarrow +\infty} E_2(t)$ exists;
- (iii) $\lim_{t \rightarrow +\infty} \Psi(x(t)) = \Psi(z).$

Proof of Lemma 2.4. Let us compute the first order time derivative of $E_2(\cdot)$. By using classical derivation chain rule, and equation (MIG), we obtain

$$\begin{aligned} \dot{E}_2(t) &= \frac{1}{\epsilon(t)} \langle \dot{x}(t), \ddot{x}(t) \rangle - \frac{\dot{\epsilon}(t)}{2\epsilon(t)^2} \|\dot{x}(t)\|^2 + \frac{1}{\epsilon(t)} \langle \nabla \Phi(x(t)), \dot{x}(t) \rangle - \frac{\dot{\epsilon}(t)}{\epsilon(t)^2} \Phi(x(t)) + \langle \nabla \Psi(x(t)), \dot{x}(t) \rangle \\ &= \frac{1}{\epsilon(t)} \langle \dot{x}(t), \ddot{x}(t) + \nabla \Phi(x(t)) + \epsilon(t) \nabla \Psi(x(t)) \rangle - \frac{\dot{\epsilon}(t)}{\epsilon(t)^2} \left(\frac{1}{2} \|\dot{x}(t)\|^2 + \Phi(x(t)) \right) \\ &= -\frac{\gamma}{\epsilon(t)} \|\dot{x}(t)\|^2 - \frac{\dot{\epsilon}(t)}{\epsilon(t)^2} \left(\frac{1}{2} \|\dot{x}(t)\|^2 + \Phi(x(t)) \right), \end{aligned}$$

which proves item (i). By assumption (\mathcal{H}_3) , there exists some positive constant k such that

$$0 \leq -\dot{\epsilon}(t) \leq k\epsilon^2(t).$$

As a consequence

$$\dot{E}_2(t) \leq k \left(\frac{1}{2} \|\dot{x}(t)\|^2 + \Phi(x(t)) \right).$$

Relying on the integrability property on $]0, +\infty[$ of $\|\dot{x}(\cdot)\|^2$ (see Lemma 2.1 (ii)), and $\Phi(x(\cdot))$ (see Lemma 2.2 (ii)), we deduce that $\dot{E}_2(\cdot)$ is majorized by an integrable function. Since E_2 is bounded from below, it follows that

$$\lim_{t \rightarrow +\infty} E_2(t) \text{ exists,}$$

that's item (ii). Since $E_2(t) \geq \Psi(x(t))$, by using Lemma 2.2 (iii) we obtain

$$\begin{aligned} \lim_{t \rightarrow +\infty} E_2(t) &\geq \liminf_{t \rightarrow +\infty} \Psi(x(t)) \\ &\geq \Psi(z). \end{aligned}$$

Let us prove that

$$\lim_{t \rightarrow +\infty} E_2(t) = \Psi(z).$$

To achieve this, we argue by contradiction and assume that

$$\lim_{t \rightarrow +\infty} E_2(t) > \Psi(z).$$

It will follow that for t sufficiently large and for some positive α

$$\frac{1}{2\epsilon(t)} \|\dot{x}(t)\|^2 + \frac{1}{\epsilon(t)} \Phi(x(t)) + \Psi(x(t)) > \Psi(z) + \alpha.$$

Hence

$$\frac{1}{2} \|\dot{x}(t)\|^2 + \Phi(x(t)) + \epsilon(t) (\Psi(x(t)) - \Psi(z)) > \alpha \epsilon(t).$$

By integration of this inequality, using the fact that $\int_0^{+\infty} \epsilon(t) dt = +\infty$ (assumption (\mathcal{H}_1)), and the fact that (equation (25))

$$\lim_{t \rightarrow +\infty} \int_0^t [\Phi(x(s)) + \epsilon(s) (\Psi(x(s)) - \Psi(z))] ds \text{ exists}$$

we obtain a contradiction. Thus

$$(28) \quad \Psi(z) = \lim_{t \rightarrow +\infty} E_2(t) \geq \limsup_{t \rightarrow +\infty} \Psi(x(t)).$$

Combining (28) with Lemma 2.2 (iii), we finally obtain item (iii)

$$\lim_{t \rightarrow +\infty} \Psi(x(t)) = \Psi(z).$$

End of the proof of Theorem 2.1.

We can now apply Opial's lemma. By Lemma 2.2 (i), for every $z \in S$ there is convergence of $\|x(\cdot) - z\|$. By Lemma 2.1 (iv) $\lim_{t \rightarrow +\infty} \Phi(x(t)) = \inf_{\mathcal{H}} \Phi = 0$. This clearly implies that every weak cluster point of $x(\cdot)$ belongs to C . Moreover by Lemma 2.4 (iii), $\lim_{t \rightarrow +\infty} \Psi(x(t)) = \Psi(z)$. This forces every weak cluster point of $x(\cdot)$ to belong to S . In the two preceding results, we use the lower semicontinuity of Φ and Ψ for the weak topology. \square

3. STRONG CONVERGENCE RESULTS

The (MIG) system was first introduced in the strongly convex case by [8] and its generalization [17]. The proof relies on a geometrical argument which uses the weak compactness of the set:

$$\{x | \langle \nabla \Psi(x), x - x_\infty \rangle \leq 0\},$$

where x_∞ is the unique point in S . Our approach provides a different hindsight of the convergence result, as follows.

Theorem 3.1. *Let us suppose the standing assumptions (\mathcal{H}_0) , and (\mathcal{H}_1) $\int_0^{+\infty} \epsilon(t) dt = +\infty$. Assume that the potential Ψ is uniformly convex, i.e., there exists a function $\theta : \mathbb{R}_+ \rightarrow \mathbb{R}_+$ such that $\theta(t_n) \rightarrow 0 \Rightarrow t_n \rightarrow 0$ and, for every x and y in H ,*

$$\langle \nabla \Psi(x) - \nabla \Psi(y), x - y \rangle \geq \theta(\|x - y\|).$$

Let x_∞ be the unique point in $S = \operatorname{argmin}\{\Psi | \operatorname{argmin} \Phi\}$. Assume one of the following:

- (i) [8, Theorem 2.3], [17, Theorem 5.1] the set $\{x | \langle \nabla \Psi(x), x - x_\infty \rangle \leq 0\}$ is weakly compact.
- (ii)

$$(\mathcal{H}_2) \quad \forall p \in \mathcal{C} \quad \int_0^{+\infty} [\Phi^*(\epsilon(t)p) - \sigma_C(\epsilon(t)p)] dt < +\infty.$$

$$(\mathcal{H}_3) \quad 0 \leq -\dot{\epsilon}(t) \leq k\epsilon^2(t) \quad \text{for some positive constant } k, \text{ and large } t.$$

Let x be a classical maximal solution of (MIG). Then $\lim_{t \rightarrow +\infty} \|x(t) - x_\infty\| = 0$

Proof of Theorem 3.1.

Case (i) We refer to [8] and its generalization [17].

Case (ii) Just follow the proof of Theorem 2.1. With x_∞ the unique point in S , Equations (15) and (16), together with the strong convexity of Ψ , yield

$$\ddot{h}_{x_\infty}(t) + \gamma \dot{h}_{x_\infty}(t) + \Phi(x(t)) + \epsilon(t) \langle \nabla \Psi(x(t)), x(t) - x_\infty \rangle \leq \|\dot{x}(t)\|^2.$$

Writing

$$\langle \nabla \Psi(x), x - x_\infty \rangle = \langle \nabla \Psi(x) - \nabla \Psi(x_\infty), x - x_\infty \rangle + \langle \nabla \Psi(x_\infty), x - x_\infty \rangle,$$

and following the proof of Lemma 2.2, we deduce

$$\ddot{h}_{x_\infty}(t) + \gamma \dot{h}_{x_\infty}(t) + \epsilon(t) \theta(\|x(t) - x_\infty\|) \leq [\Phi^*(\epsilon(t)p) - \sigma_C(\epsilon(t)p)] + \|\dot{x}(t)\|^2,$$

with $p = -\nabla \Psi(x_\infty)$ with $p \in N_C(x_\infty)$. Thus

$$\int_0^{+\infty} \epsilon(t) \theta(\|x(t) - x_\infty\|) < +\infty.$$

Once the convergence of h_{x_∞} is obtained, its limit can only be 0.

4. MULTISCALE ASPECTS

4.1. Invariance by affine rescaling. Let us show that the moderate decrease conditions are invariant by affine rescaling. In particular this means that our asymptotic analysis of (MIG) system is not affected by an affine change of variable. Given $t \mapsto x(t)$ a solution trajectory of (MIG), and $a > 0$ a positive parameter (coefficient of the affine rescaling) let us set

$$y(t) := x(at).$$

Direct application of the derivation chain rule yields

$$\ddot{y}(t) + a\gamma \dot{y}(t) + a^2 \nabla \Phi(y(t)) + a^2 \epsilon(at) \nabla \Psi(y(t)) = 0.$$

This is still a (MIG) system with the new data:

$$\tilde{\Phi}(y) = a^2 \Phi(y)$$

$$\tilde{\Psi}(y) = \Psi(y)$$

$$\tilde{\epsilon}(t) = a^2 \epsilon(at)$$

Since $\operatorname{argmin} \tilde{\Phi} = \operatorname{argmin} \Phi = C$, the cone \mathcal{C} is invariant by affine rescaling, and condition (\mathcal{H}_2) for the rescaled equation (4.1) is equivalent to

$$\forall p \in \mathcal{C} \quad \int_0^{+\infty} [\tilde{\Phi}^*(\tilde{\epsilon}(t)p) - \sigma_C(\tilde{\epsilon}(t)p)] dt < +\infty.$$

Since

$$\begin{aligned} \tilde{\Phi}^*(\tilde{\epsilon}(t)p) &= a^2 \Phi^*(\epsilon(at)p) \\ \sigma_C(\tilde{\epsilon}(t)p) &= a^2 \sigma_C(\epsilon(t)p) \end{aligned}$$

we have

$$\begin{aligned} \int_0^{+\infty} [\tilde{\Phi}^*(\tilde{\epsilon}(t)p) - \sigma_C(\tilde{\epsilon}(t)p)] dt &= a^2 \int_0^{+\infty} [\Phi^*(\epsilon(at)p) - \sigma_C(\epsilon(at)p)] dt \\ &= a \int_0^{+\infty} [\Phi^*(\epsilon(t)p) - \sigma_C(\epsilon(t)p)] dt < +\infty. \end{aligned}$$

Thus, we recover the (\mathcal{H}_2) condition on the initial sytem. Clearly conditions (\mathcal{H}_1) and (\mathcal{H}_3) are also invariant by affine rescaling.

4.2. Asymptotic penalization. In this section, by changing the time scale, we show dynamic systems that are connected and having multiscale aspects. We start from

$$(MIG) \quad m\ddot{x}(t) + \gamma\dot{x}(t) + \nabla\Phi(x(t)) + \epsilon(t)\nabla\Psi(x(t)) = 0,$$

which involves a control term which is asymptotically vanishing ($\epsilon(t) \rightarrow 0$ as $t \rightarrow +\infty$). The nonnegative (mass) parameter m has been introduced in the equation to cover both the second order and the first order system (obtained by taking $m = 0$). We use the following dictionary, introduced in [9].

Lemma 4.1 (dictionary). *Let T_β and T_ϵ be two elements in $(\mathbb{R}_+ \setminus \{0\}) \cup +\infty$. Take two functions of class C^1*

$$\begin{aligned} \beta : [0, T_\beta[&\rightarrow \mathbb{R}_+ \setminus \{0\}; \\ \epsilon : [0, T_\epsilon[&\rightarrow \mathbb{R}_+ \setminus \{0\}. \end{aligned}$$

Define $t_\beta : [0, T_\epsilon[\rightarrow [0, T_\beta[$ and $t_\epsilon : [0, T_\beta[\rightarrow [0, T_\epsilon[$ by

$$\int_0^{t_\beta(t)} \beta(s) ds = t \quad \text{and} \quad \int_0^{t_\epsilon(t)} \epsilon(s) ds = t.$$

Assume that, for every t ,

$$\epsilon(t)\beta(t_\beta(t)) = 1.$$

Then

$$\begin{aligned} t_\epsilon \circ t_\beta &= \operatorname{id}_{[0, T_\epsilon[} \quad \text{and} \quad T_\epsilon = \int_0^{T_\beta} \beta; \\ t_\beta \circ t_\epsilon &= \operatorname{id}_{[0, T_\beta[} \quad \text{and} \quad T_\beta = \int_0^{T_\epsilon} \epsilon. \end{aligned}$$

If x is a solution trajectory of

$$(MIG) \quad m\ddot{x}(t) + \gamma\dot{x}(t) + \nabla\Phi(x(t)) + \epsilon(t)\nabla\Psi(x(t)) = 0,$$

then $w := x \circ t_\epsilon$ is a solution trajectory of

$$(MIG_\beta) \quad \frac{m}{\beta(t)}\ddot{w}(t) + (\gamma - m\frac{\dot{\beta}(t)}{\beta^2(t)})\dot{w}(t) + \beta(t)\nabla\Phi(w(t)) + \nabla\Psi(w(t)) = 0.$$

Conversely, if w is a solution of (MIG_β) , then $w \circ t_\beta$ is a solution of (MIG) .

Proof of Lemma 4.1. Let us make the change of variable associated with function $t_\epsilon(\cdot)$. Note that $\dot{t}_\epsilon(t) = \beta(t)$. Let $x(\cdot)$ be a solution of (MIG) and write the system (MIG) at the point $t_\epsilon(t)$:

$$m\ddot{x}(t_\epsilon(t)) + \gamma\dot{x}(t_\epsilon(t)) + \nabla\Phi(x(t_\epsilon(t))) + \epsilon(t_\epsilon(t))\nabla\Psi(x(t_\epsilon(t))) = 0.$$

After multiplication by $\dot{t}_\epsilon(t)$ we get

$$m\dot{t}_\epsilon(t)\ddot{x}(t_\epsilon(t)) + \gamma\dot{t}_\epsilon(t)\dot{x}(t_\epsilon(t)) + \dot{t}_\epsilon(t)\nabla\Phi(x(t_\epsilon(t))) + \dot{t}_\epsilon(t)\epsilon(t_\epsilon(t))\nabla\Psi(x(t_\epsilon(t))) = 0.$$

Set $w = x \circ t_\epsilon$. According to $\dot{w}(t) = \dot{t}_\epsilon(t)\dot{x}(t_\epsilon(t))$, $\dot{t}_\epsilon(t) = \beta(t)$ and $\dot{t}_\epsilon(t)\epsilon(t_\epsilon(t)) = 1$, we obtain

$$(29) \quad m\dot{t}_\epsilon(t)\ddot{x}(t_\epsilon(t)) + \gamma\dot{w}(t) + \beta(t)\nabla\Phi(w(t)) + \nabla\Psi(w(t)) = 0.$$

From

$$\ddot{w}(t) = \dot{t}_\epsilon^2(t)\ddot{x}(t_\epsilon(t)) + \ddot{t}_\epsilon(t)\dot{x}(t_\epsilon(t)),$$

we obtain

$$(30) \quad \dot{t}_\varepsilon(t) \ddot{x}(t_\varepsilon(t)) = \frac{1}{\beta(t)} \ddot{w}(t) - \frac{\dot{\beta}(t)}{\beta^2(t)} \dot{w}(t).$$

Combining (29) and (30) gives the result. \square

Accordingly, all our results can be written for the system (MIG_β) . Let us formulate the conditions of moderate decrease of $\epsilon(\cdot)$ with the help of $\beta(\cdot)$.

$$\begin{aligned} (\mathcal{H}_1) \quad & \int_0^{+\infty} \epsilon(t) dt = +\infty \text{ becomes } \beta(t) \rightarrow +\infty \text{ as } t \rightarrow +\infty; \\ (\mathcal{H}_2) \quad & \forall p \in \mathcal{C} \quad \int_0^{+\infty} [\Phi^*(\epsilon(t)p) - \sigma_C(\epsilon(t)p)] dt < +\infty \text{ becomes } \int_0^{+\infty} \beta(t) \left[\Phi^*\left(\frac{p}{\beta(t)}\right) - \sigma_C\left(\frac{p}{\beta(t)}\right) \right] dt < +\infty; \\ (\mathcal{H}_3) \quad & 0 \leq -\dot{\epsilon}(t) \leq k\epsilon^2(t) \text{ for some positive constant } k, \text{ and large } t \text{ becomes } \dot{\beta}(t) \leq k\beta(t). \end{aligned}$$

Note that (\mathcal{H}_1) and (\mathcal{H}_3) imply $\lim_{t \rightarrow +\infty} \frac{\dot{\beta}(t)}{\beta^2(t)} = 0$. Thus (MIG_β) asymptotically behaves like

$$\frac{m}{\beta(t)} \ddot{w}(t) + \gamma \dot{w}(t) + \beta(t) \nabla \Phi(w(t)) + \nabla \Psi(w(t)) = 0.$$

Note that the coefficient $\frac{m}{\beta(t)}$ in front of the acceleration term goes to zero as time t goes to infinity. It is an interesting subject for further research to study the related system

$$\ddot{w}(t) + \gamma \dot{w}(t) + \beta(t) \nabla \Phi(w(t)) + \nabla \Psi(w(t)) = 0,$$

where the coefficient in front of the acceleration term is a fixed real positive number.

5. COUPLED GRADIENT DYNAMICS

Throughout this section we make the following assumptions:

- $\mathcal{H} = \mathcal{X}_1 \times \mathcal{X}_2$ is the cartesian product of two Hilbert spaces, set $x = (x_1, x_2)$;
- $\Phi(x) = f_1(x_1) + f_2(x_2)$, $f_1 \in \Gamma_0(\mathcal{X}_1)$, $f_2 \in \Gamma_0(\mathcal{X}_2)$ are convex functions which are continuously differentiable, ∇f_i is Lipschitz continuous on bounded sets of \mathcal{X}_i , $i = 1, 2$. We assume that $C_i = \text{argmin } f_i$ is nonempty.
- $\Psi(x) = \frac{1}{2} \|L_1 x_1 - L_2 x_2\|_{\mathcal{Z}}^2$, $L_1 \in L(\mathcal{X}_1, \mathcal{Z})$ and $L_2 \in L(\mathcal{X}_2, \mathcal{Z})$ are linear continuous operators acting respectively from \mathcal{X}_1 and \mathcal{X}_2 into a third Hilbert space \mathcal{Z} ;
- $\epsilon : \mathbb{R}^+ \rightarrow \mathbb{R}^+$ is a function of t which tends to 0 as t goes to $+\infty$.

In this setting

$$(\text{MIG}) \quad \ddot{x}(t) + \gamma \dot{x}(t) + \nabla \Phi(x(t)) + \epsilon(t) \nabla \Psi(x(t)) = 0,$$

becomes

$$(31) \quad \begin{cases} \ddot{x}_1(t) + \gamma \dot{x}_1(t) + \nabla f_1(x_1(t)) + \epsilon(t) L_1^*(L_1 x_1(t) - L_2 x_2(t)) = 0 \\ \ddot{x}_2(t) + \gamma \dot{x}_2(t) + \nabla f_2(x_2(t)) + \epsilon(t) L_2^*(L_2 x_2(t) - L_1 x_1(t)) = 0. \end{cases}$$

Because of the quadratic property of Ψ , condition (\mathcal{H}_1) can be equivalently written

$$(32) \quad \epsilon(\cdot) \in L^2(0, +\infty) \setminus L^1(0, +\infty).$$

As a straight application of Theorem 2.1, assuming (32) and the growth condition

$$0 \leq -\dot{\epsilon}(t) \leq k\epsilon^2(t) \text{ for some positive constant } k, \text{ and large } t,$$

we obtain that $x(t) = (x_1(t), x_2(t)) \rightarrow x_\infty = (x_{1,\infty}, x_{2,\infty})$ weakly in \mathcal{H} where $(x_{1,\infty}, x_{2,\infty})$ is a solution of

$$\min \{ \|L_1 x_1 - L_2 x_2\| : x_1 \in \text{argmin } f_1, x_2 \in \text{argmin } f_2 \},$$

assuming that the solution set of the above problem is nonempty. The above model describes two oscillators coupled by a force $\epsilon(t) \nabla \Psi(x(t))$ which asymptotically vanishes. When the control variable $\epsilon(t)$ tends to zero in a moderate way, i.e., $\epsilon(\cdot) \in L^2(0, +\infty) \setminus L^1(0, +\infty)$, the control forces the two oscillators to stabilize asymptotically in equilibria that are "as close as possible" to each other.

This model has a natural interpretation as a Nash equilibration process for potential games. Consider two players with respective individual payoff functions $f_1 \in \Gamma_0(\mathcal{X}_1)$ and $f_2 \in \Gamma_0(\mathcal{X}_2)$. Suppose that, at time t , the two players have a joint payoff $\frac{\epsilon(t)}{2} \|L_1 x_1 - L_2 x_2\|^2$. This attractive potential may reflect some joint constraint on resources. System

(31) describes a continuous dynamic which, at time t , is attached to the potential team game (Monderer-Shapley, 1996) associated to the payoff functions

$$\begin{cases} F_1(x_1, x_2) = f_1(x_1) + \frac{\epsilon(t)}{2} \|L_1 x_1 - L_2 x_2\|^2 \\ F_2(x_1, x_2) = f_2(x_2) + \frac{\epsilon(t)}{2} \|L_1 x_1 - L_2 x_2\|^2 \end{cases}$$

The second order nature of the dynamics, and the damping term reflects the importance of the inertia, and friction aspects in the process. This model provides an explanation of what seems strange phenomenon, why specific equilibria are observed in real life, despite the fact that no force is present who can explain. It is history, and the fact that these forces disappear asymptotically, which gives a response.

6. COUPLED WAVE EQUATIONS

Consider the following infinite dimensional version of the situation discussed above. It modelizes two wave equations, with Neumann boundary conditions, which are coupled by an asymptotic vanishing term. Let us give

- Ω a bounded open set in \mathbb{R}^n ;
- $\gamma > 0$ (damping parameter), $\alpha_1 > 0, \alpha_2 > 0$ (waves propagation speed) which are positive constants;
- $h_1, h_2 \in L^2(\Omega)$ verifying $\int_{\Omega} h_1 = \int_{\Omega} h_2 = 0$ (compatibility condition);
- $u_{01}, u_{02} \in H^1(\Omega)$, $v_{01}, v_{02} \in L^2(\Omega)$ (Cauchy data);
- $\epsilon : [0, +\infty[\rightarrow \mathbb{R}_+$ a non-increasing function of class \mathcal{C}^2 such that $\lim_{t \rightarrow +\infty} \epsilon(t) = \lim_{t \rightarrow +\infty} \dot{\epsilon}(t) = 0$.

The system is written as follows:

$$(33) \quad \begin{cases} \frac{\partial^2 u_1}{\partial t^2} + \gamma \frac{\partial u_1}{\partial t} - \alpha_1 \Delta u_1 + \epsilon(t)(u_1 - u_2) = h_1 & \text{on } \Omega \times]0, +\infty[\\ \frac{\partial^2 u_2}{\partial t^2} + \gamma \frac{\partial u_2}{\partial t} - \alpha_2 \Delta u_2 + \epsilon(t)(u_2 - u_1) = h_2 & \text{on } \Omega \times]0, +\infty[\\ \frac{\partial u_1}{\partial n} = \frac{\partial u_2}{\partial n} = 0 & \text{on } \partial\Omega \times]0, +\infty[\\ u_1(0) = u_{01}, u_2(0) = u_{02} \\ \frac{\partial u_1}{\partial t}(0) = v_{01}, \frac{\partial u_2}{\partial t}(0) = v_{02}. \end{cases}$$

Let us interpret the system (33) in a functional setting. Take

- $\mathcal{H} = L^2(\Omega) \times L^2(\Omega)$ with its Hilbert product structure.
- $\Phi : \mathcal{H} = L^2(\Omega) \times L^2(\Omega) \rightarrow \mathbb{R} \cup \{+\infty\}$, with for any $v = (v_1, v_2) \in \mathcal{H}$, $\Phi(v) = \Phi_1(v_1) + \Phi_2(v_2)$,

$$(34) \quad \Phi_i(v) = \begin{cases} \frac{\alpha_i}{2} \int_{\Omega} |\nabla v(x)|^2 dx - \int_{\Omega} h_i(x) v(x) dx & \text{if } v \in H^1(\Omega) \\ +\infty & \text{if } v \in L^2(\Omega) \setminus H^1(\Omega) \end{cases}$$

- $\Psi : \mathcal{H} = L^2(\Omega) \times L^2(\Omega) \rightarrow \mathbb{R}$, with for any $v = (v_1, v_2) \in \mathcal{H}$,

$$\Psi(v) = \frac{1}{2} \int_{\Omega} |v_1(x) - v_2(x)|^2 dx.$$

The system (33) can be equivalently formulated as the Cauchy problem for the following evolution equation

$$\ddot{u}(t) + \gamma \dot{u}(t) + \partial\Phi(u(t)) + \epsilon(t)\nabla\Psi(u(t)) = 0.$$

In the above equation, $\partial\Phi$ is the subdifferential of Φ in \mathcal{H} , which, because of the decoupled formulation of Φ , gives for each component, the (minus) Laplace operator, with homogenous Neumann boundary condition, and with respective coefficients α_1 and α_2 , see for example [5, Theorem 17.2.12]. Note that computing the gradient at u of the coupling potential Ψ gives two opposite forces $u_1 - u_2$ and $u_2 - u_1$ (action and reaction principle).

As we have already noted, Theorem 2.1 can not be directly applied to this sytem, because Φ is only lower semi-continuous. Still, our approach can be adapted to this situation. The point is that the dimension of the space is not involved in our analysis, and Rellich-Kondrakov compact embedding of $H^1(\Omega)$ into $L^2(\Omega)$, makes the functional Φ inf-compact on $\mathcal{H} = L^2(\Omega) \times L^2(\Omega)$. Thus we are naturally led to develop a similar analysis to that used in Theorem 2.1. We omit the details of the proof of convergence, which combines our proof with the arguments used in the study of the wave equation in [2, Theorem 2.1], and [8, Proposition 4.1]. Let us just describe the asymptotic selection result.

Because of the compatibility condition $\int_{\Omega} h_i = 0$, the variational problem

$$(35) \quad \min \left\{ \frac{\alpha_i}{2} \int_{\Omega} |\nabla v(x)|^2 dx - \int_{\Omega} h_i(x) v(x) dx : v \in H^1(\Omega), \int_{\Omega} v(x) dx = 0 \right\}$$

has a unique solution, which is denoted by \bar{u}_i , see for example [5, Theorem 6.2.3]. As a consequence, the solution set $C = \operatorname{argmin} \Phi$ is given by

$$(36) \quad C = \{(\bar{u}_1 + r_1, \bar{u}_2 + r_2) : r_1 \in \mathbb{R}, r_2 \in \mathbb{R}\}.$$

Because of the quadratic property of Ψ , the moderate decrease condition (\mathcal{H}_1) can be equivalently written

$$(37) \quad \epsilon(\cdot) \in L^2(0, +\infty) \setminus L^1(0, +\infty).$$

Thus, under the above condition, we obtain that the solution trajectory $t \mapsto (u_1(t), u_2(t))$ of (33) converges strongly in $\mathcal{H} = L^2(\Omega) \times L^2(\Omega)$ to some $u_{\infty} = (u_{1,\infty}, u_{2,\infty})$ which is a solution of the minimization problem

$$(38) \quad \min \left\{ \int_{\Omega} |v_1(x) - v_2(x)|^2 dx : v_i = \bar{u}_i + r_i, r_i \in \mathbb{R} \right\}.$$

An elementary calculus gives $u_{1,\infty} = \bar{u}_1 + r_1$, $u_{2,\infty} = \bar{u}_2 + r_2$, with $r_1 = r_2$. Equivalently,

$$\int_{\Omega} u_{1,\infty}(x) dx = \int_{\Omega} u_{2,\infty}(x) dx$$

which means that, at the limit, the two waves stabilize at equilibria that are as close as possible, in the sense that they have the same mean value. Convergence of the energies allows to pass from the convergence for the L^2 norm to the convergence for the H^1 norm.

This model suggests new developments. For example, it would be interesting to study the case where the two waves propagate in different domains, and the coupling occurs only on their common boundary.

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